

ON HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR STRONGLY φ_h -CONVEX FUNCTIONS

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ABSTRACT. In this paper, using functions whose derivatives absolute values are strongly φ_h -convex with modulus $c > 0$, we obtained new inequalities related to the right and left side of Hermite-Hadamard inequality by using new integral identities

1. INTRODUCTION

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g., [4], [8, p.137]). These inequalities state that if $f : I \rightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

The inequality (1.1) has evoked the interest of many mathematicians. Especially in the last three decades numerous generalizations, variants and extensions of this inequality have been obtained, to mention a few, see ([3]-[12]) and the references cited therein.

Let I be an interval in \mathbb{R} and $h : (0, 1) \rightarrow (0, \infty)$ be a given function. A function $f : I \rightarrow [0, \infty)$ is said to be h -convex if

$$(1.2) \quad f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y)$$

for all $x, y \in I$ and $t \in (0, 1)$ [23]. This notion unifies and generalizes the known classes of functions, s -convex functions, Gudunova-Levin functions and P -functions, which are obtained by putting in (1.2), $h(t) = t$, $h(t) = t^s$, $h(t) = \frac{1}{t}$, and $h(t) = 1$, respectively. Many properties of them can be found, for instance, in [6], [7], [16], [18], [19], [21], [23].

Let us consider a function $\varphi : [a, b] \rightarrow [a, b]$ where $[a, b] \subset \mathbb{R}$. Youness have defined the φ -convex functions in [17]:

Definition 1. A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be φ -convex on $[a, b]$ if for every two points $x \in [a, b]$, $y \in [a, b]$ and $t \in [0, 1]$ the following inequality holds:

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y)).$$

Obviously, if function φ is the identity, then the classical convexity is obtained from the previous definition. Many properties of the φ -convex functions can be found, for instance, in [1], [2], [17], [20], [21].

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Moreover in [2], Cristescu have presented a version Hermite-Hadamard type inequality for the φ -convex functions as follows:

Theorem 1. *If a function $f : [a, b] \rightarrow \mathbb{R}$ is φ -convex for the continuous function $\varphi : [a, b] \rightarrow [a, b]$, then*

$$(1.3) \quad f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \leq \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \leq \frac{f(\varphi(a)) + f(\varphi(b))}{2}.$$

Recall also that a function $f : I \rightarrow \mathbb{R}$ is called strongly convex with modulus $c > 0$, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2$$

for all $x, y \in I$ and $t \in (0, 1)$. Strongly convex functions have been introduced by Polyak in [13] and they play an important role in optimization theory and mathematical economics. Various properties and applicatins of them can be found in the literature see ([13]-[16]) and the references cited therein.

In [22], Sarikaya have introduced the following notion of the strongly φ_h -convex functions with modulus $c > 0$, and give some properties of them:

A function $f : D \rightarrow [0, \infty)$ is said to be strongly φ_h -convex with modulus $c > 0$, if

$$(1.4) \quad \begin{aligned} & f(t\varphi(x) + (1-t)\varphi(y)) \\ & \leq h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)) - ct(1-t)(\varphi(x) - \varphi(y))^2 \end{aligned}$$

for all $x, y \in D$ and $t \in (0, 1)$. In particular, if f satisfies (1.4) with $h(t) = t$, $h(t) = t^s$ ($s \in (0, 1)$), $h(t) = \frac{1}{t}$, and $h(t) = 1$, then f is said to be strongly φ -convex, strongly φ_s -convex, strongly φ -Gudunova-Levin function and strongly φ - P -function, respectively. The notion of φ_h -convex function corresponds to the case $c \rightarrow 0$.

In this article, using functions whose derivatives absolute values are strongly φ_h -convex with modulus $c > 0$, we obtained new inequalities releted to the right and left side of Hermite-Hadamard inequality by using new integral identities. In particular if $\varphi = 0$ is taken as, our results obtained reduce to the Hermite-Hadamard type inequality for classical convex functions.

2. MAIN RESULTS

In order to prove our main results, we establish a important integral identity as follows:

Lemma 1. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I^0 where $a, b \in I$ with $a < b$ and $\varphi : [a, b] \rightarrow [a, b]$. If f' is a Lebesgue integrable function, then the*

following equality holds;

$$\begin{aligned}
 (2.1) \quad & \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \\
 &= \frac{(\varphi(b) - \varphi(a))}{2} \int_0^1 (2t - 1) [f'(t\varphi(b) + (1-t)\varphi(a)) + ct(1-t)(\varphi(b) - \varphi(a))^2] dt.
 \end{aligned}$$

Proof. By integration by parts, we can state:

$$\begin{aligned}
 I &= \int_0^1 (2t - 1) [f'(t\varphi(b) + (1-t)\varphi(a)) + ct(1-t)(\varphi(b) - \varphi(a))^2] dt \\
 &= (2t - 1) \frac{f(t\varphi(b) + (1-t)\varphi(a))}{(\varphi(b) - \varphi(a))} \Big|_0^1 - \frac{2}{(\varphi(b) - \varphi(a))^2} \int_0^1 f(t\varphi(b) + (1-t)\varphi(a)) dt \\
 &= \frac{f(\varphi(b)) + f(\varphi(a))}{(\varphi(b) - \varphi(a))} - \frac{2}{(\varphi(b) - \varphi(a))} \int_0^1 f(t\varphi(b) + (1-t)\varphi(a)) dt.
 \end{aligned}$$

Using the change of the variable $x = t\varphi(b) + (1-t)\varphi(a)$ for $t \in [0, 1]$, which gives

$$(2.2) \quad I = \frac{f(\varphi(b)) + f(\varphi(a))}{(\varphi(b) - \varphi(a))} - \frac{2}{(\varphi(b) - \varphi(a))^2} \int_{\varphi(a)}^{\varphi(b)} f(x) dx.$$

Multiplying the both sides of (2.2) by $\frac{(\varphi(b) - \varphi(a))}{2}$, we obtain

$$\frac{(\varphi(b) - \varphi(a))}{2} I = \frac{f(\varphi(b)) + f(\varphi(a))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx$$

which is required. \square

Theorem 2. Let $h : (0, 1) \rightarrow (0, \infty)$ be a given function. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$ and Lebesgue integrable function. If $|f'|$ is strongly φ_h -convex with respect to $c > 0$ for the continuous function $\varphi : [a, b] \rightarrow [a, b]$ and $\varphi(a) < \varphi(b)$, then the following inequality holds;

$$\begin{aligned}
 (2.3) \quad & \left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right| \\
 & \leq \frac{(\varphi(b) - \varphi(a))}{2} [|f'(\varphi(b))| + |f'(\varphi(a))|] \int_0^1 |2t - 1| h(t) dt
 \end{aligned}$$

for all $t \in (0, 1)$.

Proof. From Lemma 1 and by using strongly φ_h -convexity functions with modulus $c > 0$ of $|f'|$, we have

$$\begin{aligned}
& \left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right| \\
& \leq \frac{(\varphi(b) - \varphi(a))}{2} \int_0^1 |2t - 1| [|f'(t\varphi(b) + (1-t)\varphi(a))| + ct(1-t)(\varphi(b) - \varphi(a))^2] dt \\
& \leq \frac{(\varphi(b) - \varphi(a))}{2} \int_0^1 |2t - 1| [h(t)|f'(\varphi(b))| + h(1-t)|f'(\varphi(a))|] dt \\
& = \frac{(\varphi(b) - \varphi(a))}{2} [|f'(\varphi(b))| + |f'(\varphi(a))|] \int_0^1 |2t - 1| h(t) dt
\end{aligned}$$

where using the fact that

$$\int_0^1 h(t) dt = \int_0^1 h(1-t) dt$$

which completes the proof. \square

The following inequalities are associated the right side of Hermite-Hadamard type inequalities for strongly φ -convex, strongly φ_s -convex strongly $\varphi - P$ -convex with respect to $c > 0$, respectively.

Corollary 1. *Under the assumptions of Theorem 2 with $h(t) = t$, $t \in (0, 1)$, we have*

$$(2.4) \quad \left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right| \leq (\varphi(b) - \varphi(a)) \left(\frac{|f'(\varphi(b))| + |f'(\varphi(a))|}{8} \right).$$

Corollary 2. *Under the assumptions of Theorem 2 with $h(t) = t^s$ ($s \in (0, 1)$), $t \in (0, 1)$, we have*

$$\begin{aligned}
(2.5) \quad & \left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right| \\
& \leq \frac{(\varphi(b) - \varphi(a))}{2} \left(s + \frac{1}{2^{s+1}} \right) \left(\frac{|f'(\varphi(b))| + |f'(\varphi(a))|}{(s+1)(s+2)} \right).
\end{aligned}$$

Corollary 3. *Under the assumptions of Theorem 2 with $h(t) = 1$, $t \in (0, 1)$, we have*

$$(2.6) \quad \left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right| \leq \frac{(\varphi(b) - \varphi(a))}{2} \left(\frac{|f'(\varphi(b))| + |f'(\varphi(a))|}{2} \right).$$

Remark 1. (a) *In the case $c \rightarrow 0$ and $\varphi(x) = x$ for all $x \in [a, b]$, then inequality (2.4) coincide with the right sides of Hermite-Hadamard inequality proved by Dragomir and Agarwal in ([5]).*

(b) *In the case $c \rightarrow 0$ and $\varphi(x) = x$ for all $x \in [a, b]$, then inequality (2.5) gives the right sides of Hermite-Hadamard inequality proved by Dragomir and Agarwal in ([5]).*

Theorem 3. *Let $h : (0, 1) \rightarrow (0, \infty)$ be a given function. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I^0 where $a, b \in I$ with $a < b$ and Lebesgue integrable function. If $|f'|^q$ is strongly φ_h -convex with respect to $c > 0$ for the continuous function $\varphi : [a, b] \rightarrow [a, b]$, $\varphi(a) < \varphi(b)$, and*

$$A = c^q(\varphi(b) - \varphi(a))^{2q} B(q+1, q+1) - \frac{c}{6}(\varphi(b) - \varphi(a))^2 > 0,$$

then the following inequality holds;

$$\begin{aligned} & \left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right| \\ & \leq \frac{(\varphi(b) - \varphi(a))}{2^{\frac{1}{q}}} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left((|f'(\varphi(b))|^q + |f'(\varphi(a))|^q) \left(\int_0^1 h(t) dt \right) + A \right)^{\frac{1}{q}} \end{aligned}$$

for all $t \in (0, 1)$, $q \geq 1$ where B is a beta function.

Proof. From Lemma 1 and by using Hölder's integral inequality, we have

$$\begin{aligned} & \left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right| \\ & \leq \frac{(\varphi(b) - \varphi(a))}{2} \left(\int_0^1 |2t - 1|^p dt \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_0^1 [|f'(t\varphi(b) + (1-t)\varphi(a))| + ct(1-t)(\varphi(b) - \varphi(a))^2]^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f'|^q$ is strongly φ_h -convex on $[a, b]$ and using the following inequality

$$(u + v)^q \leq 2^{q-1}(u^q + v^q), \quad u, v > 0, \quad q > 1,$$

we get

$$\begin{aligned}
& \left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right| \\
& \leq \frac{(\varphi(b) - \varphi(a))}{2^{\frac{1}{q}}} \left(\int_0^1 |2t - 1|^p dt \right)^{\frac{1}{p}} \\
& \quad \times \left(\int_0^1 \left[|f'(t\varphi(b) + (1-t)\varphi(a))|^q + (ct(1-t)(\varphi(b) - \varphi(a))^2)^q \right] dt \right)^{\frac{1}{q}} \\
& \leq \frac{(\varphi(b) - \varphi(a))}{2^{\frac{1}{q}}} \left(\int_0^1 |2t - 1|^p dt \right)^{\frac{1}{p}} \\
& \quad \times \left(\int_0^1 (h(t) |f'(\varphi(b))|^q + h(1-t) |f'(\varphi(a))|^q \right. \\
& \quad \left. - ct(1-t)(\varphi(b) - \varphi(a))^2 + [ct(1-t)(\varphi(b) - \varphi(a))^2]^q) dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Thus, with simple calculations we obtain

$$\begin{aligned}
& \int_0^1 |2t - 1|^p dt = \frac{1}{p+1}, \\
& \int_0^1 t(1-t) dt = \frac{1}{6}, \quad \int_0^1 t^q(1-t)^q dt = B(q+1, q+1)
\end{aligned}$$

and

$$\int_0^1 h(t) dt = \int_0^1 h(1-t) dt.$$

Therefore, we obtain

$$\begin{aligned}
& \left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right| \\
& \leq \frac{(\varphi(b) - \varphi(a))}{2^{\frac{1}{q}}} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left((|f'(\varphi(b))|^q + |f'(\varphi(a))|^q) \left(\int_0^1 h(t) dt \right) \right. \\
& \quad \left. + c^q (\varphi(b) - \varphi(a))^{2q} B(q+1, q+1) - \frac{c}{6} (\varphi(b) - \varphi(a))^2 \right)^{\frac{1}{q}}
\end{aligned}$$

which completes the proof. \square

The following inequalities are associated the right side of Hermite-Hadamard type inequalities for strongly φ -convex, strongly φ_s -convex strongly $\varphi - P$ -convex with respect to $c > 0$, respectively.

Corollary 4. *Under the assumptions of Theorem 3 with $h(t) = t$, $t \in (0, 1)$, we have*

$$(2.7) \quad \left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right| \\ \leq \frac{(\varphi(b) - \varphi(a))}{2^{\frac{1}{q}}} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{|f'(\varphi(b))|^q + |f'(\varphi(a))|^q}{2} + A \right)^{\frac{1}{q}}.$$

Corollary 5. *Under the assumptions of Theorem 3 with $h(t) = t^s$ ($s \in (0, 1)$), $t \in (0, 1)$, we have*

$$(2.8) \quad \left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right| \\ \leq \frac{(\varphi(b) - \varphi(a))}{2^{\frac{1}{q}}} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{|f'(\varphi(b))|^q + |f'(\varphi(a))|^q}{s+1} + A \right)^{\frac{1}{q}}.$$

Corollary 6. *Under the assumptions of Theorem 3 with $h(t) = 1$, $t \in (0, 1)$, we have*

$$(2.9) \quad \left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right| \\ \leq \frac{(\varphi(b) - \varphi(a))}{2^{\frac{1}{q}}} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} (|f'(\varphi(b))|^q + |f'(\varphi(a))|^q + A)^{\frac{1}{q}}.$$

Remark 2. *In the case $c \rightarrow 0$, inequalities (2.7) (2.8) and (2.9) reduce the right sides of Hermite-Hadamard type inequality for φ -convex, φ_s -convex and $\varphi - P$ -convex functions, respectively.*

Lemma 2. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I^0 where $a, b \in I$ with $a < b$ and $\varphi : [a, b] \rightarrow [a, b]$. If f' is Lebesgue integrable function, then the following equality holds;*

$$\frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \\ = (\varphi(b) - \varphi(a)) \left\{ \int_0^{\frac{1}{2}} t [f'(t\varphi(a) + (1-t)\varphi(b)) + ct(1-t)(\varphi(b) - \varphi(a))^2] dt \right. \\ \left. \int_{\frac{1}{2}}^1 (t-1) [f'(t\varphi(a) + (1-t)\varphi(b)) + ct(1-t)(\varphi(b) - \varphi(a))^2] dt \right\}.$$

Proof. By integration by parts, we can state:

$$\begin{aligned}
J_1 &= \int_0^{\frac{1}{2}} t [f'(t\varphi(a) + (1-t)\varphi(b)) + ct(1-t)(\varphi(b) - \varphi(a))^2] dt \\
&= t \frac{f(t\varphi(a) + (1-t)\varphi(b))}{(\varphi(a) - \varphi(b))} \Big|_0^{\frac{1}{2}} \\
(2.10) \quad &- \frac{1}{(\varphi(a) - \varphi(b))} \int_0^{\frac{1}{2}} f(t\varphi(a) + (1-t)\varphi(b)) dt + c(\varphi(b) - \varphi(a))^2 \int_0^{\frac{1}{2}} t^2(1-t) dt \\
&= \frac{1}{2(\varphi(a) - \varphi(b))} f\left(\frac{\varphi(b) + \varphi(a)}{2}\right) \\
&\quad + \frac{1}{(\varphi(b) - \varphi(a))} \int_0^{\frac{1}{2}} f(t\varphi(a) + (1-t)\varphi(b)) dt + \frac{5c}{3 \times 2^6} (\varphi(b) - \varphi(a))^2.
\end{aligned}$$

and similarly

$$\begin{aligned}
J_2 &= \int_{\frac{1}{2}}^1 (t-1) [f'(t\varphi(a) + (1-t)\varphi(b)) + ct(1-t)(\varphi(b) - \varphi(a))^2] dt \\
&= (t-1) \frac{f(t\varphi(a) + (1-t)\varphi(b))}{(\varphi(a) - \varphi(b))} \Big|_{\frac{1}{2}}^1 \\
(2.11) \quad &- \frac{1}{(\varphi(a) - \varphi(b))} \int_{\frac{1}{2}}^1 f(t\varphi(a) + (1-t)\varphi(b)) dt - c(\varphi(b) - \varphi(a))^2 \int_{\frac{1}{2}}^1 t(1-t)^2 dt \\
&= \frac{1}{2(\varphi(a) - \varphi(b))} f\left(\frac{\varphi(b) + \varphi(a)}{2}\right) \\
&\quad + \frac{1}{(\varphi(b) - \varphi(a))} \int_{\frac{1}{2}}^1 f(t\varphi(a) + (1-t)\varphi(b)) dt - \frac{5c}{3 \times 2^6} (\varphi(b) - \varphi(a))^2.
\end{aligned}$$

Adding (2.10) and (2.11) and rewritting, we easily deduce

(2.12)

$$J = J_1 + J_2 = \frac{1}{(\varphi(b) - \varphi(a))} \int_0^1 f(t\varphi(a) + (1-t)\varphi(b)) dt - \frac{1}{(\varphi(b) - \varphi(a))} f\left(\frac{\varphi(b) + \varphi(a)}{2}\right).$$

Using the change of the variable $x = t\varphi(a) + (1-t)\varphi(b)$ for $t \in [0, 1]$, and multiplying the both sides of (2.12) by $(\varphi(b) - \varphi(a))$, we obtain

$$(\varphi(b) - \varphi(a))J = \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right)$$

which is required. \square

Theorem 4. Let $h : (0, 1) \rightarrow (0, \infty)$ be a given function. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$ and Lebesgue integrable function. If $|f'|$ is strongly φ_h -convex with respect to $c > 0$ for the continuous function $\varphi : [a, b] \rightarrow [a, b]$ and $\varphi(a) < \varphi(b)$, then the following inequality holds;

$$\begin{aligned} & \left| \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \\ & \leq (\varphi(b) - \varphi(a)) (|f'(\varphi(a))| + |f'(\varphi(b))|) \left(\int_0^{\frac{1}{2}} t [h(t) + h(1-t)] dt \right) \end{aligned}$$

for all $t \in (0, 1)$.

Proof. From Lemma 2 and by using strongly φ_h -convexity functions with modulus $c > 0$ of $|f'|$, we have

$$\begin{aligned} & \left| \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \\ & \leq (\varphi(b) - \varphi(a)) \left\{ \int_0^{\frac{1}{2}} t [h(t) |f'(\varphi(a))| + h(1-t) |f'(\varphi(b))|] dt \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 (1-t) [h(t) |f'(\varphi(a))| + h(1-t) |f'(\varphi(b))|] dt \right\} \\ & = (\varphi(b) - \varphi(a)) (|f'(\varphi(a))| + |f'(\varphi(b))|) \left(\int_0^{\frac{1}{2}} t [h(t) + h(1-t)] dt \right) \end{aligned}$$

where using the fact that

$$\int_0^{\frac{1}{2}} th(t)dt = \int_{\frac{1}{2}}^1 (1-t) h(1-t)dt$$

and

$$\int_0^{\frac{1}{2}} th(1-t)dt = \int_{\frac{1}{2}}^1 (1-t)h(t)dt$$

which completes the proof. \square

The following inequalities are associated the left side of Hermite-Hadamard type inequalities for strongly φ -convex, strongly φ_s -convex strongly $\varphi - P$ -convex with respect to $c > 0$, respectively.

Corollary 7. *Under the assumptions of Theorem 4 with $h(t) = t$, $t \in (0, 1)$, we have*

$$(2.13) \quad \left| \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right|$$

$$\leq (\varphi(b) - \varphi(a)) \left(\frac{|f'(\varphi(a))| + |f'(\varphi(b))|}{8} \right).$$

Corollary 8. *Under the assumptions of Theorem 4 with $h(t) = t^s$ ($s \in (0, 1)$), $t \in (0, 1)$, we have*

$$(2.14) \quad \left| \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right|$$

$$\leq (\varphi(b) - \varphi(a)) \left(1 + \frac{s+3}{2^{s+2}} \right) \left(\frac{|f'(\varphi(a))| + |f'(\varphi(b))|}{(s+1)(s+2)} \right).$$

Corollary 9. *Under the assumptions of Theorem 4 with $h(t) = 1$, $t \in (0, 1)$, we have*

$$(2.15) \quad \left| \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right|$$

$$\leq (\varphi(b) - \varphi(a)) \left(\frac{|f'(\varphi(a))| + |f'(\varphi(b))|}{4} \right).$$

Remark 3. (a) In the case $c \rightarrow 0$ and $\varphi(x) = x$ for all $x \in [a, b]$, then inequality (2.13) gives the right sides of Hermite-Hadamard inequality proved by Kirmaci in ([10]).

(b) In the case $c \rightarrow 0$ and $\varphi(x) = x$ for all $x \in [a, b]$, then inequality (2.14) coincide with the right sides of Hermite-Hadamard inequality proved by Dragomir and Agarwal in ([5]).

Theorem 5. Let $h : (0, 1) \rightarrow (0, \infty)$ be a given function. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I^0 where $a, b \in I$ with $a < b$ and Lebesgue integrable

function. If $|f'|^q$ is strongly φ_h -convex with respect to $c > 0$ for the continuous function $\varphi : [a, b] \rightarrow [a, b]$, $\varphi(a) < \varphi(b)$, and

$$G = c(\varphi(b) - \varphi(a))^{2q} B_{\frac{1}{2}}(q+1, q+1) - \frac{c}{12}(\varphi(b) - \varphi(a))^2 > 0,$$

then the following inequality holds;

$$\begin{aligned} & \left| \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \\ & \leq \frac{(\varphi(b) - \varphi(a))}{2^{\frac{1}{q}}} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \left(\int_0^{\frac{1}{2}} (h(t) |f'(\varphi(a))|^q + h(1-t) |f'(\varphi(b))|^q) dt + G \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_{\frac{1}{2}}^1 (h(t) |f'(\varphi(a))|^q + h(1-t) |f'(\varphi(b))|^q) dt + G \right)^{\frac{1}{q}} \right\} \end{aligned}$$

for all $t \in (0, 1)$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ where $B_r(.,.)$ is incomplete beta function.

Proof. From Lemma 2 and by using the Hölder's inequality, we have

$$\begin{aligned} & \left| \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \\ & \leq (\varphi(b) - \varphi(a)) \\ & \quad \times \left\{ \left(\int_0^{\frac{1}{2}} t^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} [f'(t\varphi(a) + (1-t)\varphi(b)) + ct(1-t)(\varphi(b) - \varphi(a))^2]^q dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_{\frac{1}{2}}^1 (t-1)^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 [f'(t\varphi(a) + (1-t)\varphi(b)) + ct(1-t)(\varphi(b) - \varphi(a))^2]^q dt \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Since $|f'|^q$ is strongly φ_h -convex on $[a, b]$ and using the following inequality

$$(u + v)^q \leq 2^{q-1}(u^q + v^q), \quad u, v > 0, \quad q > 1,$$

we get

$$\begin{aligned}
& \left| \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f\left(\frac{\varphi(b) + \varphi(a)}{2}\right) \right| \\
& \leq 2^{1-\frac{1}{q}} (\varphi(b) - \varphi(a)) \left\{ \left(\int_0^{\frac{1}{2}} t^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left[|f'(t\varphi(a) + (1-t)\varphi(b))|^q + (ct(1-t)(\varphi(b) - \varphi(a))^2)^q \right] dt \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \left(\int_{\frac{1}{2}}^1 (1-t)^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left[|f'(t\varphi(a) + (1-t)\varphi(b))|^q + (ct(1-t)(\varphi(b) - \varphi(a))^2)^q \right] dt \right)^{\frac{1}{q}} \right\} \\
& \leq \frac{(\varphi(b) - \varphi(a))}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \\
& \quad \times \left\{ \left(\int_0^{\frac{1}{2}} \left[h(t) |f'(\varphi(a))|^q + h(1-t) |f'(\varphi(b))|^q \right. \right. \right. \\
& \quad \left. \left. \left. - ct(1-t)(\varphi(b) - \varphi(a))^2 + (ct(1-t)(\varphi(b) - \varphi(a))^2)^q \right] dt \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \left(\int_{\frac{1}{2}}^1 \left[h(t) |f'(\varphi(a))|^q + h(1-t) |f'(\varphi(b))|^q \right. \right. \right. \\
& \quad \left. \left. \left. - ct(1-t)(\varphi(b) - \varphi(a))^2 + (ct(1-t)(\varphi(b) - \varphi(a))^2)^q \right] dt \right)^{\frac{1}{q}} \right\} \\
& = \frac{(\varphi(b) - \varphi(a))}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \\
& \quad \times \left\{ \left(\int_0^{\frac{1}{2}} \left(h(t) |f'(\varphi(a))|^q + h(1-t) |f'(\varphi(b))|^q \right) dt \right. \right. \\
& \quad \left. \left. - \frac{c}{12} (\varphi(b) - \varphi(a))^2 + c(\varphi(b) - \varphi(a))^{2q} B_{\frac{1}{2}}(q+1, q+1) \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \left(\int_{\frac{1}{2}}^1 \left(h(t) |f'(\varphi(a))|^q + h(1-t) |f'(\varphi(b))|^q \right) dt \right. \right. \\
& \quad \left. \left. - \frac{c}{12} (\varphi(b) - \varphi(a))^2 + c(\varphi(b) - \varphi(a))^{2q} B_{\frac{1}{2}}(q+1, q+1) \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Thus, with simple calculations we obtain

$$\int_0^{\frac{1}{2}} t^p dt = \int_{\frac{1}{2}}^1 (1-t)^p dt = \frac{1}{2^{p+1}(p+1)},$$

$$\int_0^{\frac{1}{2}} t(1-t)dt = \int_{\frac{1}{2}}^1 t(1-t)dt = \frac{1}{12},$$

$$\int_0^{\frac{1}{2}} t^q(1-t)^q dt = \int_{\frac{1}{2}}^1 t^q(1-t)^q dt = B_{\frac{1}{2}}(q+1, q+1).$$

Therefore, using the above obtained results, we have

$$\begin{aligned} & \left| \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \\ & \leq \frac{(\varphi(b) - \varphi(a))}{2} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\left[\int_0^{\frac{1}{2}} (h(t) |f'(\varphi(a))|^q + h(1-t) |f'(\varphi(b))|^q) dt + G \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\int_{\frac{1}{2}}^1 (h(t) |f'(\varphi(a))|^q + h(1-t) |f'(\varphi(b))|^q) dt + G \right]^{\frac{1}{q}} \right) \end{aligned}$$

which completes the proof. \square

The following inequalities are associated the left side of Hermite-Hadamard type inequalities for strongly φ -convex, strongly φ_s -convex strongly $\varphi - P$ -convex with respect to $c > 0$, respectively.

Corollary 10. *Under the assumptions of Theorem 5 with $h(t) = t$, $t \in (0, 1)$, we have*

$$\begin{aligned} & \left| \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \\ & \leq (2.16) \frac{(\varphi(b) - \varphi(a))}{2^{\frac{1}{q}}} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \\ & \quad \times \left\{ \left(\frac{|f'(\varphi(a))|^q + 3|f'(\varphi(b))|^q}{8} + G \right)^{\frac{1}{q}} + \left(\frac{3|f'(\varphi(a))|^q + |f'(\varphi(b))|^q}{8} + G \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Corollary 11. *Under the assumptions of Theorem 5 with $h(t) = t^s$ ($s \in (0, 1)$), $t \in (0, 1)$, we have*

$$\begin{aligned}
 (2.17) \quad & \left| \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \\
 & \leq \frac{(\varphi(b) - \varphi(a))}{2^{\frac{1}{q}}} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \\
 & \times \left\{ \left(\frac{1}{2^{s+1}(s+1)} |f'(\varphi(a))|^q + \frac{1}{(s+1)} \left(1 - \frac{1}{2^{s+1}}\right) |f'(\varphi(b))|^q + G \right)^{\frac{1}{q}} \right. \\
 & \left. + \left(\frac{1}{(s+1)} \left(1 - \frac{1}{2^{s+1}}\right) |f'(\varphi(a))|^q + \frac{1}{2^{s+1}(s+1)} |f'(\varphi(b))|^q + G \right)^{\frac{1}{q}} \right\}.
 \end{aligned}$$

Corollary 12. *Under the assumptions of Theorem 5 with $h(t) = 1$, $t \in (0, 1)$, we have*

$$\begin{aligned}
 (2.18) \quad & \left| \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \\
 & \leq \frac{(\varphi(b) - \varphi(a))}{2^{\frac{1}{q}-1}} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{|f'(\varphi(b))|^q + |f'(\varphi(a))|^q}{2} + G \right)^{\frac{1}{q}}.
 \end{aligned}$$

Remark 4. *In the case $c \rightarrow 0$, inequalities (2.16) (2.17) and (2.18) reduce the right sides of Hermite-Hadamard type inequality for φ -convex, φ_s -convex and φ - P -convex functions, respectively.*

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